

On the phenomenology of neutrino oscillations in vacuum

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Abstract

A simple method of the calculation of neutrino transition probabilities in vacuum in the general case of n massive neutrinos is presented. The method proposed fully utilizes the unitarity of the mixing matrix. Three-neutrino case for both neutrino mass hierarchies is considered in some details. Transitions in the case of the sterile neutrinos are also discussed.

1 Introduction

The observation of neutrino oscillations in atmospheric Super-Kamiokande [1], solar SNO [2], reactor KamLAND [3] and other neutrino experiments [4] is one of the most important recent discovery in the particle physics.

Neutrino oscillations are based on the assumption that states of flavor neutrinos ν_e, ν_μ, ν_τ and sterile neutrinos $\nu_{s1}, \nu_{s2}, \dots$ are described by the coherent superpositions of the states of neutrinos with definite masses (see, for example, [5], [6], [7], [8],[9])

$$|\nu_\alpha\rangle = \sum_{i=1}^n U_{\alpha i}^* |\nu_i\rangle. \quad (1)$$

Here $n = 3 + n_s$ (n_s is the number of sterile neutrinos), U is an unitary $n \times n$ mixing matrix, $|\nu_i\rangle$ is left-handed state of neutrino with mass m_i . The states $|\nu_\alpha\rangle$ satisfy the condition

$$\langle \nu_{\alpha'} | \nu_\alpha \rangle = \delta_{\alpha' \alpha}. \quad (2)$$

If at $t = 0$ the flavor neutrino ν_α is produced at the time t we have

$$|\nu_\alpha\rangle_t = \sum_{\alpha'} |\nu_{\alpha'}\rangle \langle \nu_{\alpha'} | e^{-iH_0 t} | \nu_\alpha \rangle, \quad (3)$$

where H_0 is the free Hamiltonian.

Thus, the probability of the transition $\nu_\alpha \rightarrow \nu_{\alpha'}$ during time t is given by the expression

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = |\langle \nu_{\alpha'} | e^{-iH_0 t} | \nu_\alpha \rangle|^2 = \left| \sum_i \langle \nu_{\alpha'} | \nu_i \rangle e^{-iE_i t} \langle \nu_i | \nu_\alpha \rangle \right|^2 = \left| \sum_i U_{\alpha'i} e^{-iE_i t} U_{\alpha i}^* \right|^2. \quad (4)$$

It is obvious that the factor $U_{\alpha i}^*$ is the amplitude of the transition from initial flavor state into the state with definite mass $|\nu_i\rangle$, the factor $e^{-iE_i t}$ describes propagation in this state and the factor $U_{\alpha'i}$ is the amplitude of the transition from the state with definite mass into the final state $|\nu_{\alpha'}\rangle$. For the ultrarelativistic neutrino we have

$$E_i = \sqrt{p^2 + m_i^2} \simeq E + \frac{m_i^2}{2E}, \quad (5)$$

where $E = p$ is the energy of neutrino at $m_i \rightarrow 0$.

The expression (4) can be rewritten in the form

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \left| \sum_i U_{\alpha'i} e^{-2i\Delta_{pi}} U_{\alpha i}^* \right|^2. \quad (6)$$

Here p is an arbitrary fixed index,

$$\Delta_{pi} = (E_i - E_p)t \simeq \frac{\Delta m_{pi}^2 L}{4E}, \quad \Delta m_{pi}^2 = m_i^2 - m_p^2. \quad (7)$$

We used the relation

$$t = L, \quad (8)$$

where L is the source-detector distance¹

From (6) it is obvious that transitions between different neutrinos are possible if two conditions are satisfied

1. At least one neutrino mass-squared difference is different from zero.
2. Neutrinos are mixed ($U \neq 1$).

¹Let us notice that the validity of the relation (8) was confirmed in the high-accuracy recent OPERA measurement [10]

2 Standard expression for the transition probability

From (6) we obviously have

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \sum_i |U_{\alpha'i}|^2 |U_{\alpha i}|^2 + 2 \operatorname{Re} \sum_{i>k} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} e^{-2i\Delta_{ki}}. \quad (9)$$

From the unitarity of the matrix U ($\sum_i U_{\alpha'i} U_{\alpha i}^* = \delta_{\alpha'\alpha}$) we find

$$\sum_i |U_{\alpha'i}|^2 |U_{\alpha i}|^2 + 2 \operatorname{Re} \sum_{i>k} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} = \delta_{\alpha'\alpha} \quad (10)$$

From (9) and (10) for the transition probability we obtain the following expression

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \delta_{\alpha'\alpha} - 2 \operatorname{Re} \sum_{i>k} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} (1 - e^{-2i\Delta_{ki}}). \quad (11)$$

Finally we obtain the following standard expression for the probability of neutrino transition in vacuum (see [8], [11])

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \delta_{\alpha'\alpha} - 4 \sum_{i>k} \operatorname{Re} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} \sin^2 \Delta_{ki} + 2 \sum_{i>k} \operatorname{Im} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} \sin 2\Delta_{ki} \quad (12)$$

In order to obtain $\bar{\nu}_\alpha \rightarrow \bar{\nu}_{\alpha'}$ transition probability we need to make in (12) the change $U_{\alpha i} \rightarrow U_{\alpha i}^*$. Thus, we have

$$P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_{\alpha'}) = \delta_{\alpha'\alpha} - 4 \sum_{i>k} \operatorname{Re} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} \sin^2 \Delta_{ki} - 2 \sum_{i>k} \operatorname{Im} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} \sin 2\Delta_{ki} \quad (13)$$

It is obvious that in the case of $\alpha' = \alpha$ the last terms of (12) and (13) are equal to zero. We have

$$P(\nu_\alpha \rightarrow \nu_\alpha) = P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\alpha). \quad (14)$$

This relation is a consequence of the CPT invariance.

If CP invariance in the lepton sector holds, in this case $U_{\alpha i} = U_{\alpha i}^*$ and

$$P(\nu_\alpha \rightarrow \nu_{\alpha'}) = P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_{\alpha'}) \quad \alpha' \neq \alpha. \quad (15)$$

From (12) and (13) for the CP asymmetry we have

$$A_{\alpha'\alpha}^{CP} = P(\nu_\alpha \rightarrow \nu_{\alpha'}) - P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_{\alpha'}) = 4 \sum_{i>k} A_{\alpha'\alpha}^{ik} \sin 2\Delta_{ki}. \quad (16)$$

where

$$A_{\alpha'\alpha}^{ik} = \text{Im } U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k}. \quad (17)$$

It obvious from (17) that

$$A_{\alpha'\alpha}^{ik} = -A_{\alpha'\alpha}^{ki}. \quad (18)$$

In the case of the three-neutrino mixing the CP asymmetry is characterized by $A_{\alpha'\alpha}^{21}$, $A_{\alpha'\alpha}^{31}$ and $A_{\alpha'\alpha}^{32}$. We will show now that (see [12])

$$A_{\alpha'\alpha}^{21} = -A_{\alpha'\alpha}^{31} = A_{\alpha'\alpha}^{32}. \quad (19)$$

In fact, taking into account the unitarity of the mixing matrix we have

$$\sum_k A_{\alpha'\alpha}^{ik} = 0. \quad (20)$$

From (18) and (20) we find

$$A_{\alpha'\alpha}^{12} + A_{\alpha'\alpha}^{13} = 0, \quad A_{\alpha'\alpha}^{21} + A_{\alpha'\alpha}^{23} = 0, \quad A_{\alpha'\alpha}^{31} + A_{\alpha'\alpha}^{32} = 0 \quad (21)$$

From these equations we easily find the relations (19). Thus, in the three-neutrino case the CP asymmetry has the form

$$A_{\alpha'\alpha}^{CP} = 4 A_{\alpha'\alpha}^{21} (\sin 2\Delta_{12} + \sin 2\Delta_{23} - \sin 2\Delta_{13}) \quad (22)$$

where

$$\Delta_{13} = \Delta_{12} + \Delta_{23}. \quad (23)$$

For any a and b we have the relation

$$\sin a + \sin b - \sin(a+b) - 4 \sin \frac{a+b}{2} \sin \frac{a}{2} \sin \frac{b}{2}. \quad (24)$$

Thus, the CP asymmetry in the three-neutrino case is given by the expression

$$A_{\alpha'\alpha}^{CP} = 16 A_{\alpha'\alpha}^{21} \sin(\Delta_{12} + \Delta_{23}) \sin \Delta_{12} \sin \Delta_{23}. \quad (25)$$

3 Alternative way of the calculation of the transition probability

We will present here a simple method of the calculation of the probability of the neutrino transition in vacuum. In expression for the vacuum transition probability, presented below, the unitarity of the mixing matrix will be fully utilized. In particular, the expression (25) for the CP asymmetry will be obtained directly from the general expression for the transition probability without any additional calculations.

Let us return back to the expression (6). Taking into account the unitarity of the mixing matrix we can rewrite this expression in the form

$$\begin{aligned} P(\nu_\alpha \rightarrow \nu_{\alpha'}) &= |\delta_{\alpha'\alpha} + \sum_i U_{\alpha'i} (e^{-2i\Delta_{pi}} - 1) U_{\alpha i}^*|^2 \\ &= |\delta_{\alpha'\alpha} - 2i \sum_i U_{\alpha'i} U_{\alpha i}^* e^{-i\Delta_{pi}} \sin \Delta_{pi}|^2 \end{aligned} \quad (26)$$

It is obvious that the index i in (26) runs over values $i \neq p$.

From (26) we have

$$\begin{aligned} P(\nu_\alpha \rightarrow \nu_{\alpha'}) &= \delta_{\alpha'\alpha} - 4 \sum_i |U_{\alpha i}|^2 \sin^2 \Delta_{pi} \delta_{\alpha'\alpha} + 4 \sum_i |U_{\alpha'i}|^2 |U_{\alpha i}|^2 \sin^2 \Delta_{pi} \\ &+ 8 \operatorname{Re} \sum_{i>k} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} e^{-i(\Delta_{pi} - \Delta_{pk})} \sin \Delta_{pi} \sin \Delta_{pk} \end{aligned} \quad (27)$$

Finally, we find the following general expression for $\nu_\alpha \rightarrow \nu_{\alpha'}$ ($\bar{\nu}_\alpha \rightarrow \bar{\nu}_{\alpha'}$) transition probability

$$\begin{aligned} P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_{\alpha'}) &= \delta_{\alpha'\alpha} - 4 \sum_i |U_{\alpha i}|^2 (\delta_{\alpha'\alpha} - |U_{\alpha'i}|^2) \sin^2 \Delta_{pi} \\ &+ 8 \sum_{i>k} \operatorname{Re} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} \cos(\Delta_{pi} - \Delta_{pk}) \sin \Delta_{pi} \sin \Delta_{pk} \\ &\pm 8 \sum_{i>k} \operatorname{Im} U_{\alpha'i} U_{\alpha i}^* U_{\alpha'k}^* U_{\alpha k} \sin(\Delta_{pi} - \Delta_{pk}) \sin \Delta_{pi} \sin \Delta_{pk} \end{aligned} \quad (28)$$

4 Three-neutrino oscillations

4.1 General expressions for $\nu_l \rightarrow \nu_{l'}$ ($\bar{\nu}_l \rightarrow \bar{\nu}_{l'}$) transition probabilities

In the case of the three-neutrino mixing there are two independent mass-squared differences. From analysis of neutrino oscillation data it follows that one mass-squared difference is much smaller than the other one. Correspondingly, two three-neutrino mass spectra are possible

1. Normal hierarchy (NH)

$$m_1 < m_2 < m_3, \quad \Delta m_{12}^2 \ll \Delta m_{23}^2. \quad (29)$$

2. Inverted hierarchy (IH)²

$$m_3 < m_1 < m_2, \quad \Delta m_{12}^2 \ll |\Delta m_{13}^2|. \quad (30)$$

Let us denote two independent neutrino mass-squared differences Δm_S^2 (solar) and Δm_A^2 (atmospheric). We have

$$\Delta m_{12}^2 = \Delta m_S^2, \quad m_{23}^2 = \Delta m_A^2 \quad (NH) \quad \Delta m_{12}^2 = \Delta m_S^2, \quad \Delta m_{13}^2 = -\Delta m_A^2 \quad (IH). \quad (31)$$

In the case of the NH it is natural to choose $p = 2$. From the general expression (28) we have

$$\begin{aligned} P^{NH}(\bar{\nu}_l \rightarrow \bar{\nu}_{l'}) &= \delta_{ll'} - 4 \sum_i |U_{li}|^2 (\delta_{ll'} - |U_{l'i}|^2) \sin^2 \Delta_S \\ &- 4 \sum_i |U_{li}|^2 (\delta_{ll'} - |U_{l'i}|^2) \sin^2 \Delta_A - 8 \operatorname{Re} U_{l'3} U_{l3}^* U_{l'1}^* U_{l1} \cos(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S \\ &\mp 8 \operatorname{Im} U_{l'3} U_{l3}^* U_{l'1}^* U_{l1} \sin(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S \end{aligned} \quad (32)$$

²Notice that neutrino masses are labeled differently in the case of NH and IH. This allows to introduce the same mixing angles in both cases

In the case of the IH we choose $p = 1$. For the transition probability we obtain the following expression

$$\begin{aligned}
P^{IH}(\bar{\nu}_l \rightarrow \bar{\nu}_{l'}) &= \delta_{ll'} - 4 \sum_i |U_{l2}|^2 (\delta_{ll'} - |U_{l'2}|^2) \sin^2 \Delta_S \\
&- 4 \sum_i |U_{l3}|^2 (\delta_{ll'} - |U_{l'3}|^2) \sin^2 \Delta_A - 8 \operatorname{Re} U_{l'3} U_{l3}^* U_{l'2}^* U_{l2} \cos(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S \\
&\pm 8 \sum_{i>k} \operatorname{Im} U_{l'3} U_{l3}^* U_{l'2}^* U_{l2} \sin(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S
\end{aligned} \tag{33}$$

The expressions (32) and (33) differ by the change $U_{l1} \rightarrow U_{l2}$ and by the sign of the last term. Notice that for the CP asymmetry directly from (32) and (33) we have

$$A_{l'l}^{CP} = -16 \operatorname{Im} U_{l'3} U_{l3}^* U_{l'1}^* U_{l1} \sin(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S \tag{34}$$

in the case of NH and

$$A_{l'l}^{CP} = 16 \operatorname{Im} U_{l'3} U_{l3}^* U_{l'1}^* U_{l1} \sin(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S \tag{35}$$

in the case of IH. In the standard parameterizations the 3×3 PMNS [13],[14] mixing matrix U is characterized by three mixing angles and one CP phase and has the form

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix}. \tag{36}$$

Here $c_{12} = \cos \theta_{12}$, $s_{12} = \sin \theta_{12}$ etc.

4.2 Leading approximation

From analysis of the neutrino oscillation data it follows that two neutrino oscillation parameters are small:

$$\frac{\Delta m_S^2}{\Delta m_A^2} \simeq 3 \cdot 10^{-2}, \quad \sin^2 \theta_{13} \simeq 2.4 \cdot 10^{-2} \tag{37}$$

In atmospheric region of the parameter $\frac{L}{E}$ ($\frac{\Delta m_A^2 L}{2E} \gtrsim 1$) effects of neutrino oscillations are large. In the first, leading approximation we can neglect

small contributions of Δm_S^2 and $\sin^2 \theta_{13}$ into neutrino transition probabilities. From (32), (33) and (36) for the probability of ν_μ ($\bar{\nu}_\mu$) to survive (for both neutrino mass spectra) we obtain the following expression

$$\begin{aligned} P^{NH}(\bar{\nu}_l \rightarrow \bar{\nu}_{l'}) &\simeq P^{IH}(\bar{\nu}_l \rightarrow \bar{\nu}_{l'}) \simeq 1 - 4|U_{\mu 3}|^2(1 - |U_{\mu 3}|^2) \sin^2 \Delta m_A^2 \frac{L}{4E} \\ &= 1 - \sin^2 2\theta_{23} \sin^2 \Delta m_A^2 \frac{L}{4E}. \end{aligned} \quad (38)$$

In the leading approximation we have $P(\nu_\mu \rightarrow \nu_e) \simeq 0$ and

$$P(\nu_\mu \rightarrow \nu_\tau) \simeq 1 - P(\nu_\mu \rightarrow \nu_\mu) \simeq \sin^2 2\theta_{23} \sin^2 \Delta m_A^2 \frac{L}{4E}. \quad (39)$$

Thus, in the atmospheric region predominantly two-neutrino $\nu_\mu \rightleftharpoons \nu_\tau$ oscillations take place.

Let us consider now $\bar{\nu}_e \rightarrow \bar{\nu}_e$ transition in the reactor Kamland region ($\frac{\Delta m_S^2 L}{2E} \gtrsim 1$). Neglecting contribution of $\sin^2 \theta_{13}$ we have

$$P^{NH}(\bar{\nu}_e \rightarrow \bar{\nu}_e) \simeq P^{IH}(\bar{\nu}_e \rightarrow \bar{\nu}_e) \simeq 1 - \sin^2 2\theta_{12} \sin^2 \Delta m_S^2 \frac{L}{4E}. \quad (40)$$

For appearance probabilities we find

$$P^{NH}(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) \simeq P^{IH}(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) \simeq \sin^2 2\theta_{12} \cos^2 \theta_{23} \sin^2 \Delta m_S^2 \frac{L}{4E} \quad (41)$$

and

$$P^{NH}(\bar{\nu}_e \rightarrow \bar{\nu}_\tau) \simeq P^{IH}(\bar{\nu}_e \rightarrow \bar{\nu}_\tau) \simeq \sin^2 2\theta_{12} \sin^2 \theta_{23} \sin^2 \Delta m_S^2 \frac{L}{4E} \quad (42)$$

We have

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) - P(\bar{\nu}_e \rightarrow \bar{\nu}_\tau) \quad (43)$$

and

$$\frac{P(\bar{\nu}_e \rightarrow \bar{\nu}_\tau)}{P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu)} \simeq \tan^2 \theta_{23} \simeq 1. \quad (44)$$

Thus, in the reactor Kamland region $\bar{\nu}_e \rightleftharpoons \bar{\nu}_\mu$ and $\bar{\nu}_e \rightleftharpoons \bar{\nu}_\tau$ oscillations take place.

The expressions (38) and (38) were used for analysis of the first Super-Kamiokande atmospheric data, K2K and MINOS accelerator data and data of the reactor KamLAND experiment. Now with improved accuracy of the neutrino oscillation experiments it is more common to perform more complicated three-neutrino analysis of the data.

4.2.1 $\bar{\nu}_e \rightarrow \bar{\nu}_e$ survival probability

From (32) and (33) we can easily obtain exact three-neutrino expressions for $\bar{\nu}_e \rightarrow \bar{\nu}_e$ survival probabilities for both neutrino mass spectra. We have, correspondingly,

$$\begin{aligned} P^{\text{NH}}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = & 1 - 4 |U_{e3}|^2(1 - |U_{e3}|^2) \sin^2 \frac{\Delta m_A^2 L}{4E} \\ & - 4 |U_{e1}|^2(1 - |U_{e1}|^2) \sin^2 \frac{\Delta m_S^2 L}{4E} \\ & - 8 |U_{e3}|^2 |U_{e1}|^2 \cos \frac{(\Delta m_A^2 + \Delta m_S^2)L}{4E} \sin \frac{\Delta m_A^2 L}{4E} \sin \frac{\Delta m_S^2 L}{4E}. \end{aligned} \quad (45)$$

and

$$\begin{aligned} P^{\text{IH}}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = & 1 - 4 |U_{e3}|^2(1 - |U_{e3}|^2) \sin^2 \frac{\Delta m_A^2 L}{4E} \\ & - 4 |U_{e2}|^2(1 - |U_{e2}|^2) \sin^2 \frac{\Delta m_S^2 L}{4E} \\ & - 8 |U_{e3}|^2 |U_{e2}|^2 \cos \frac{(\Delta m_A^2 + \Delta m_S^2)L}{4E} \sin \frac{\Delta m_A^2 L}{4E} \sin \frac{\Delta m_S^2 L}{4E}. \end{aligned} \quad (46)$$

In the standard parameterization of the PMNS mixing matrix we have

$$\begin{aligned} P^{\text{NS}}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = & 1 - \sin^2 2\theta_{13} \sin^2 \frac{\Delta m_A^2 L}{2E} \\ & - (\cos^2 \theta_{13} \sin^2 2\theta_{12} + \sin^2 2\theta_{13} \cos^4 \theta_{12}) \sin^2 \frac{\Delta m_S^2 L}{2E} \\ & - 2 \sin^2 2\theta_{13} \cos^2 \theta_{12} \cos \frac{(\Delta m_A^2 + \Delta m_S^2)L}{4E} \sin \frac{\Delta m_A^2 L}{4E} \sin \frac{\Delta m_S^2 L}{4E}. \end{aligned} \quad (47)$$

and

$$\begin{aligned} P^{\text{IS}}(\bar{\nu}_e \rightarrow \bar{\nu}_e) = & 1 - \sin^2 2\theta_{13} \sin^2 \frac{\Delta m_A^2 L}{2E} \\ & - (\cos^2 \theta_{13} \sin^2 2\theta_{12} + \sin^2 2\theta_{13} \sin^4 \theta_{12}) \sin^2 \frac{\Delta m_S^2 L}{2E} \\ & - 2 \sin^2 2\theta_{13} \sin^2 \theta_{12} \cos \frac{(\Delta m_A^2 + \Delta m_S^2)L}{4E} \sin \frac{\Delta m_A^2 L}{4E} \sin \frac{\Delta m_S^2 L}{4E}. \end{aligned} \quad (48)$$

4.2.2 $\nu_\mu \rightarrow \nu_e$ ($\bar{\nu}_\mu \rightarrow \bar{\nu}_e$) transition probabilities

From (32) and (33) we obtain the following expressions for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ vacuum transition probabilities:

$$\begin{aligned}
P^{\text{NS}}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) &= 4 |U_{e3}|^2 |U_{\mu3}|^2 \sin^2 \Delta_A \\
&+ 4 |U_{e1}|^2 |U_{\mu1}|^2 \sin^2 \Delta_S \\
&- 8 \operatorname{Re} U_{e3} U_{\mu3}^* U_{e1}^* U_{\mu1} \cos(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S \\
&\mp 8 \operatorname{Im} U_{e3} U_{\mu3}^* U_{e1}^* U_{\mu1} \sin(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S.
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
P^{\text{IS}}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) &= 4 |U_{e3}|^2 |U_{\mu3}|^2 \sin^2 \Delta_A \\
&+ 4 |U_{e2}|^2 |U_{\mu2}|^2 \sin^2 \Delta_S \\
&- 8 \operatorname{Re} U_{e3} U_{\mu3}^* U_{e2}^* U_{\mu2} \cos(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S \\
&\pm 8 \operatorname{Im} U_{e3} U_{\mu3}^* U_{e2}^* U_{\mu2} \sin(\Delta_A + \Delta_S) \sin \Delta_A \sin \Delta_S.
\end{aligned} \tag{50}$$

Using the standard parameterization of the PMNS mixing matrix in the case of NH we have

$$\begin{aligned}
P^{\text{NS}}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) &= \sin^2 2\theta_{13} s_{23}^2 \sin^2 \frac{\Delta m_A^2 L}{4E} \\
&+ (\sin^2 2\theta_{12} c_{13}^2 c_{23}^2 + \sin^2 2\theta_{13} c_{12}^4 s_{23}^2 + K c_{12}^2 \cos \delta) \sin^2 \frac{\Delta m_S^2 L}{4E} \\
&+ (2 \sin^2 2\theta_{13} s_{23}^2 c_{12}^2 + K \cos \delta) \cos \frac{(\Delta m_A^2 + \Delta m_S^2) L}{4E} \sin \frac{\Delta m_A^2 L}{4E} \sin \frac{\Delta m_S^2 L}{4E} \\
&\mp K \sin \delta \sin \frac{(\Delta m_A^2 + \Delta m_S^2) L}{4E} \sin \frac{\Delta m_A^2 L}{4E} \sin \frac{\Delta m_S^2 L}{4E}.
\end{aligned} \tag{51}$$

Here

$$K = \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} c_{13}. \tag{52}$$

In the case of the inverted neutrino mass hierarchy we find the following expressions for the transition probabilities

$$\begin{aligned}
P^{\text{IS}}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) &= \sin^2 2\theta_{13}s_{23}^2 \sin^2 \frac{\Delta m_A^2 L}{4E} \\
&+ (\sin^2 2\theta_{12}c_{13}^2c_{23}^2 + \sin^2 2\theta_{13}s_{12}^4s_{23}^2 - Ks_{12}^2 \cos \delta) \sin^2 \frac{\Delta m_S^2 L}{4E} \\
&+ (2 \sin^2 2\theta_{13}s_{23}^2s_{12}^2 - K \cos \delta) \cos \frac{(\Delta m_A^2 + \Delta m_S^2)L}{4E} \sin \frac{\Delta m_A^2 L}{4E} \sin \frac{\Delta m_S^2 L}{4E} \\
&\mp K \sin \delta \sin \frac{(\Delta m_A^2 + \Delta m_S^2)L}{4E} \sin \frac{\Delta m_A^2 L}{4E} \sin \frac{\Delta m_S^2 L}{4E}. \tag{53}
\end{aligned}$$

Formulas (51) and (53) can be used for analysis of the data of T2K long baseline accelerator experiment in which matter effects are negligible.

5 Transitions in the case of sterile neutrinos

From existing data some indications in favor of neutrino oscillations driven by "large" ($\sim 1\text{eV}^2$) neutrino mass-squared difference(s) were obtained.(see [15], [16]). These data (if correct) would proof existence of sterile neutrinos.

From general formula (28) we can easily obtain transition probabilities in such cases. In the framework of mixing of four massive neutrinos we will consider first the simplest 3+1 scheme in which the forth mass is separated from three close masses by a $\sim 1\text{eV}$ gap. Let us choose $p = 1$. In the region of $\frac{L}{E}$ sensitive to large neutrino mass-squared difference ($\frac{\Delta m^2 L}{4E} \gtrsim 1$, $\Delta m^2 \equiv \Delta m_{14}^2$) we have $\Delta_{1i} \simeq 0$, $i = 2, 3$. From (28) we find in this case

$$P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_{\alpha'}) \simeq \delta_{\alpha'\alpha} - 4|U_{\alpha 4}|^2(\delta_{\alpha'\alpha} - |U_{\alpha' 4}|^2) \sin^2 \frac{\Delta m^2 L}{4E}. \tag{54}$$

In the case of reactor antineutrinos we have

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - 4|U_{e4}|^2(1 - |U_{e4}|^2) \sin^2 \frac{\Delta m^2 L}{4E} \tag{55}$$

For $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ transition (LSND) we find

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = 4|U_{e4}|^2|U_{\mu 4}|^2 \sin^2 \frac{\Delta m^2 L}{4E}. \tag{56}$$

Let us consider now 3+2 scheme with 2 masses m_4 and m_5 separated from three close masses by a $\sim 1\text{eV}$ gaps. Let us choose $p = 1$. In the region of $\frac{L}{E}$ sensitive to Δm_{14}^2 and Δm_{15}^2 we have $\Delta_{1i} \simeq 0$, $i = 2, 3$. From (28) we find in this case

$$\begin{aligned}
P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) &= 4|U_{e4}|^2|U_{\mu4}|^2 \sin^2 \frac{\Delta m_{14}^2 L}{4E} + 4|U_{e5}|^2|U_{\mu5}|^2 \sin^2 \frac{\Delta m_{15}^2 L}{4E} \\
&+ 8 \operatorname{Re} U_{e5} U_{\mu5}^* U_{e4}^* U_{\mu4} \cos\left(\frac{\Delta m_{15}^2 L}{4E} - \frac{\Delta m_{14}^2 L}{4E}\right) \sin \frac{\Delta m_{15}^2 L}{4E} \sin \frac{\Delta m_{14}^2 L}{4E} \\
&\pm 8 \operatorname{Im} U_{e5} U_{\mu5}^* U_{e4}^* U_{\mu4} \sin\left(\frac{\Delta m_{15}^2 L}{4E} - \frac{\Delta m_{14}^2 L}{4E}\right) \sin \frac{\Delta m_{15}^2 L}{4E} \sin \frac{\Delta m_{14}^2 L}{4E} \quad (57)
\end{aligned}$$

For $\bar{\nu}_\alpha \rightarrow \bar{\nu}_\alpha$ survival probability we find the following expression

$$\begin{aligned}
P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\alpha) &= 1 - 4|U_{\alpha4}|^2(1 - |U_{\alpha4}|^2) \sin \frac{\Delta m_{14}^2 L}{4E} - 4|U_{\alpha5}|^2(1 - |U_{\alpha5}|^2) \sin \frac{\Delta m_{15}^2 L}{4E} \\
&+ 8 |U_{\alpha5}|^2|U_{\alpha4}|^2 \cos\left(\frac{\Delta m_{15}^2 L}{4E} - \frac{\Delta m_{14}^2 L}{4E}\right) \sin \frac{\Delta m_{15}^2 L}{4E} \sin \frac{\Delta m_{14}^2 L}{4E} \quad (58)
\end{aligned}$$

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